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Tensor operators and projection techniques in infinite dimensional representations of semi-simple Lie algebras

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Abstract. Polynomial identities satisfied by the infinitesimal generators of a semi-simple Lie group are employed to construct projection operators which project a tensor product representation $V(\lambda)\otimes V$ ($V(\lambda)$ finite dimensional and irreducible, V an infinite dimensional representation admitting an infinitesimal character) onto a primary sub-representation. Such projection operators project out the generalised shift components of a tensor operator acting on an infinite dimensional representation V in direct analogy with the finite dimensional analysis of Bracken and Green (1971). Applications of these methods to evaluating the matrix elements of the generators of the locally compact groups $\mathrm{U}(p, q)$ and $\mathrm{O}(p, q)$ in unitary (discrete) irreducible representations are discussed.

1. Introduction

Characteristic identities satisfied by the infinitesimal generators of a semi-simple Lie group have been studied by a number of authors (see Gould (1982) and references quoted therein). Such identities have been shown to be a powerful tool for the analysis of finite dimensional representations of the group. The work of Bracken and Green (1971) and Green (1971) demonstrates how such identities may be used to construct projection operators which may be applied to a systematic analysis of vector operators on finite dimensional representations of the group. In fact it was recognised early by Fano (1962) (see also Baird and Biedenharn 1964), at least for the lower order unitary groups, that polynomial identities satisfied by the infinitesimal generators of the unitary groups were useful, particularly for the task of discussing generalised Wigner and Racah coefficients. This idea of Fano's was extended in Gould (1980, 1981) using the techniques of Bracken and Green (1971) and Green (1971) to obtain raising and lowering operators and the fundamental Wigner coefficients for the Lie groups $\mathrm{O}(n)$ and $\mathrm{U}(n)$. Moreover, extensions of this technique to arbitrary semi-simple (compact) Lie Groups were given. More recently the techniques of Bracken and Green have been extended to give a solution to the Clebsch–Gordan multiplicity problem (Edwards and Gould 1982) and the equivalent problem of labelling all tensor operators (in finite dimensional representations) for a semi-simple Lie algebra.

It is the aim of this paper to extend the techniques of Bracken and Green (1971) and Green (1971) to infinite dimensional representations of a semi-simple Lie algebra. It has been shown (see Gould 1982 and Kostant 1975) that the characteristic identities for semi-simple Lie algebras also hold in infinite dimensional representations. Using these identities it be shown that on an arbitrary representation admitting an infinitesimal character (herein called characteristic representations) one may construct projection operators analogous to the projection operators of Green (1971) and Bracken and Green (1971) in finite dimensions. However, in infinite dimensional representations, the projection operators take a more complicated form due to the fact that one does not have complete reducibility in infinite dimensions. These projection operators may then be applied, as in the finite dimensional case, to project out the shift components of a tensor operator acting on an infinite dimensional characteristic representation. However, since infinite dimensional (characteristic) representations need not possess maximal (or minimal) weights care must be taken in interpreting the nature of a shift tensor operator. To this end we aim to set up the framework for discussion of tensor operators and their shift components on infinite dimensional characteristic representations.

Our work is directly related to the nature of the tensor product space $V(\lambda)\otimes V$ where $V(\lambda)$ is a finite dimensional irreducible representation and V is an infinite dimensional characteristic representation. This includes the cases where V is a Verma module, which has been treated by Bernstein *et al* 1971 (see also Dixmier 1977), and the more general case where V is a Harish-Chandra module which has been treated by Kostant (1975), Zuckerman (1977) and Klimyk (1977) (see also references quoted therein). The projection operators of this paper are in fact an explicit construction of the projection functors appearing implicitly in the work of Zuckerman (1977). Such projection operators project the tensor product space $V(\lambda)\otimes V$ onto a primary sub-representation.

Such considerations are important for an analysis of tensor operators on unitary representations of locally compact groups such as $\mathrm{O}(p, q)$ and $\mathrm{U}(p, q)$. This includes the important problem of finding the matrix elements of the group generators in the discrete series of unitary representations. Our work demonstrates that such an analysis is possible in direct analogy with the finite dimensional case.

Finally we remark that our methods also apply to the classical Lie super-algebras. It has been demonstrated by Jarvis and Green (1979) that the characteristic identities hold also for the finite dimensional representations of the Lie super-algebras. For this case, as in the infinite dimensional analysis of this paper, one does not have complete reducibility for the finite dimensional representations. However, the methods of this paper may also be applied to construct projection operators which may be used to project out generalised shift components (generalised in a sense to be discussed in this paper) of tensor operators. In particular this opens up the possibility of a complete determination of the matrix elements of the Lie super-algebra generators, in finite dimensional irreducible representations, along the lines suggested in Gould (1980, 1981)

2. Fundamentals

Our notation follows that of Humphreys (1972) and Gould (1982). Let L be a complex semi-simple Lie algebra of rank l, let U be the universal enveloping algebra of L, and let Z be the centre of U. Select a Cartan sub-algebra H of L, with dual space H^*, and let Φ denote the set of roots of L relative to H. Let Φ^+ denote the system of positive roots and take δ to be half the sum of the positive roots. Finally let $\Lambda^+\subset H^*$ be the set of dominant integral linear functions on H and let W (resp. W) denote the Weyl group (resp. translated Weyl group).

We define an infinitesimal character χ as an algebra homomorphism of Z into the scalars $\mathbb{C}$. If $z_1, \ldots, z_l \in Z$ are algebraically independent, then a character is uniquely determined by the scalars $\chi(z_i)$ which may be arbitrary complex numbers.

If M is a representation of the Lie algebra L then M also gives a representation of the universal enveloping algebra which is a (Noetherian) associative algebra. Hence, we adopt the language of Ring Theory, and refer to a representation of the Lie algebra L as a module over U.

We say that a module M over U admits an infinitesimal character if the elements of the centre Z take constant values on M. Such a module determines an algebra homorphism

$$\chi_M : Z \to \mathbb{C}, \qquad z \to \chi_M(z),$$

where $\chi_M(z)$ is the eigenvalue of the central element z on M. In such a case we say that M admits the infinitesimal character χ_M. If v_0 is a maximal weight vector of weight $\lambda \in H^*$ then v_0 determines an algebra homomorphism

$$\chi_\lambda : Z \to \mathbb{C}$$

where $\chi_\lambda(z)$ is the eigenvalue of $z \in Z$ on v_0. The characters χ_λ play a fundamental role in character analysis since it is a theorem of Harish-Chandra (see Humphreys 1972) that every infinitesimal character χ over Z is of the form $\chi = \chi_\lambda$ for some $\lambda \in H^*$.

If M is a U-module admitting an infinitesimal character χ_λ, $\lambda \in H^*$, let us agree to call M a characteristic module and refer to χ_λ as the characteristic of the module M.

It is important to note, however, that the infinitesimal character χ_λ does not characterise the weight λ uniquely. One in fact has the following result due to Harish-Chandra.

Theorem. $\chi_\lambda = \chi_\mu$ if and only if λ and μ are $\tilde{W}$-conjugate: i.e. $\lambda = \sigma(\mu + \delta) - \delta$ for some $\sigma \in W$.

We now introduce the concept of a primary module. Before proceeding we establish some notation.

It is a well known result due to Harish-Chandra that the centre Z of U is generated as an algebra by l algebraically independent invariants $z_1, \ldots, z_l$. In polynomial algebra notation we write

$$Z = \mathbb{C}[z_1, \ldots, z_l].$$

Now associated with each infinitesimal character $\chi_\lambda : Z \to \mathbb{C}$, $\lambda \in H^*$, is its kernel I_λ which is a maximal ideal of Z of co-dimension 1. It is easily shown that I_λ is generated as an ideal by the central elements $z_i - \chi_\lambda(z_i)$ $(i = 1, \ldots, l)$, i.e.

$$I_\lambda = \sum_{i=1}^{l} Z(z_i - \chi_\lambda(z_i)).$$

Let us denote the l-dimensional vector space spanned by the $z_i - \chi_\lambda(z_i)$ $(i = 1, \ldots, l)$ by N_λ. With this convention we may write

$$I_\lambda = ZN_\lambda.$$

More generally let N_λ^m $(m \in \mathbb{Z}^+)$ denote the subspace of Z consisting of all homogeneous polynomials of degree m in the invariants $z_i - \chi_\lambda(z_i)$ $(i = 1, \ldots, l)$. Then the mth power of I_λ may be written (cf Kostant 1975)

$$I_\lambda^m = ZN_\lambda^m.$$

Definition. We call a U-module, M, a primary module of characteristic $\chi_\lambda (\lambda \in H^*)$ if there exists $m \in \mathbb{Z}^+$ such that

$$I_\lambda^m M = (0).$$

The smallest such positive integer is called the characteristic length of the module M.

Clearly characteristic modules are equivalent to primary modules of characteristic length 1. Note also that in finite dimensions, due to complete reducibility, primary modules and characteristic modules are equivalent (i.e. in finite dimensions one can only have primary modules of characteristic length 1). This is certainly not the case in infinite dimensional representations.

Suppose M is a primary module of characteristic χ_λ and characteristic length $m \geqslant 1$. We introduce the submodules

$$M_r = \{w \in M | uw = 0, \qquad \forall u \in I_\lambda^{m-r}\} \qquad r = 0, \ldots, m.$$

Then we have the (strict) descending chain of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_m = (0)$$

which we call the characteristic series of M. Now consider the factor modules M_i/M_{i+1} $(i = 0, \ldots, m-1)$. Clearly we have

$$I_\lambda (M_i/M_{i+1}) = (0),$$

since if $w \in M_i$ then $uw \in M_{i+1}$ for all $u \in I_\lambda$. Thus each factor module M_i/M_{i+1} is a characteristic module with characteristic χ_λ. This shows that primary modules are natural generalisations of characteristic modules.

Finally we assume that all modules possess a countable basis. If M is a U-module we say that the infinitesimal character χ_λ occurs in M if there exist submodules $M \supseteq M_1 \supset M_2$ such that the factor module M_1/M_2 is a characteristic module with characteristic χ_λ. Clearly if M is a primary module of characteristic χ_λ then the only infinitesimal character occurring in M is χ_λ.

3. Construction of projection operators

Let $V(\lambda)$ be a finite dimensional irreducible U-module with highest weight $\lambda \in \Lambda^+$, let π_λ be the representation afforded by $V(\lambda)$ and let $\lambda_1, \ldots, \lambda_k$ be the *distinct* weights occurring in $V(\lambda)$. Following Gould (1982) and Konstant (1975) let us consider the map

$$\partial : U \to [\text{End } V(\lambda) \otimes U]$$

defined for $x \in L$ by

$$\partial(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes x,$$

which we extend to an algebra homomorphism to all of U. Let $z \in Z$ be an arbitrary (non-trivial) central element and put

$$\tilde{z} = -\tfrac{1}{2}[\partial(z) - \pi_\lambda(z) \otimes 1 - 1 \otimes z].$$

Then, as shown in Gould (1982), $\tilde{z}$ may be viewed as a $d\times d$ matrix ($d=\dim V(\lambda)$) with entries from U.

Now let V be a (possibly infinite dimensional) characteristic U-module with characteristic $\chi_\mu(\mu\in H^*)$ and let π be the representation afforded by V. Acting on the U-module V the entries of the matrix $\tilde{z}$ become operators on V (i.e. elements of End V) whilst the matrix $\tilde{z}$ itself may be viewed as an operator on the space $V(\lambda)\otimes V$, namely

$$\tilde{z}\equiv -\tfrac{1}{2}[\pi_\lambda\otimes\pi(z)-\pi_\lambda(z)\otimes 1\otimes\pi(z)].$$

Following Gould (1982) and Kostant (1975) one knows that the matrix $\tilde{z}$ satisfies the following polynomial identity on the space V

$$\prod_{i=1}^{k}(\tilde{z}-f_{z,i}(\mu))=0, \tag{3.1}$$

where $f_{z,i}$ denotes the polynomial function on H^* defined by

$$f_{z,i}(\mu)=-\tfrac{1}{2}[\chi_{\mu+\lambda_i}(z)-\chi_\lambda(z)-\chi_\mu(z)], \qquad \mu\in H^*.$$

Clearly the numbers $f_{z,i}(\mu)$ are the generalised eigenvalues of $\tilde{z}$ on the space $V(\lambda)\otimes V$.

As a result of the polynomial identity (3.1) one sees that the infinitesimal characters occurring in the space $V(\lambda)\otimes V$ must be of the form $\chi_{\mu+\lambda_i}$ (see Kostant (1975) Theorem (5.3)). Note however that although the weights $\lambda_1,\ldots,\lambda_k$ are all distinct it is not necessarily true that the infinitesimal characters

$$\chi_{\mu+\lambda_1},\ldots,\chi_{\mu+\lambda_k} \tag{3.2}$$

are all distinct. Assume the number of distinct ones is $n\leqslant k$ and that the weights $\lambda_1,\ldots,\lambda_k$ are numbered so that the infinitesimal characters $\chi_{\mu+\lambda_i}$, $i=1,\ldots,n$, are all distinct. Let m_i be the multiplicity of the infinitesimal character $\chi_{\mu+\lambda_i}$ $(i=1,\ldots,n)$ occurring in the sequence (3.2). With this convention the polynomial identity (3.1) may be written

$$\prod_{i=1}^{n}(\tilde{z}-\alpha_i)^{m_i}=0, \tag{3.3}$$

where

$$\alpha_i=f_{z,i}(\mu).$$

Remark. The polynomial identity (3.3) may not be the minimum polynomial identity satisfied by $\tilde{z}$ on the space $V(\lambda)\otimes V$.

For ease of notation let us put $Y=V(\lambda)\otimes V$. As a result of equation (3.3) it easily follows that Y may be decomposed into a direct sum of primary submodules

$$Y=\bigoplus_{i=1}^{n} Y_i, \tag{3.4}$$

where

$$Y_i=\{y\in Y\,|\,uy=0 \qquad \text{for all } u\in I_{\mu+\lambda_i}^{m_i}\}.$$

A proof of this result for the case where V is a Harish–Chandra module is given in the paper by Kostant (1975, see Theorem 5.4). His method of proof is applicable to the general case and will not be reproduced here.

By definition each Y_i occurring in (3.4) satisfies

$$I^{m_i}_{\mu+\lambda_i} Y_i = (0),$$

so that each Y_i is a primary module of characteristic $\chi_{\mu+\lambda_i}$. If n_i denotes characteristic length of Y_i then we must have $n_i \leqslant m_i$. Also one sees that if z is such that the numbers $\chi_{\mu+\lambda_i}(z)$ $(i=1, \ldots, n)$ are all distinct then the minimum polynomial identity satisfied by $\tilde{z}$ over Y is

$$\prod_{i=1}^{n} (\tilde{z}-\alpha_i)^{n_i} = 0.$$

We see from this that a knowledge of the minimum polynomial identity satisfied by $\tilde{z}$ on Y is equivalent to a knowledge of the characteristic lengths of the primary modules Y_i.

From now on we assume that $z \in Z$ is such that the numbers $\chi_{\mu+\lambda_i}(z)$, $i=1, \ldots, n$, are all distinct. If $z_1, \ldots, z_l \in Z$ are algebraically independent then such a z may be chosen from the linear span of $z_1, \ldots, z_l$. Note that the primary modules Y_i occurring in the decomposition (3.4) are given by

$$Y_i = \{y \in Y | (z - \chi_{\mu+\lambda_i}(z))^{m_i} y = 0\}. \tag{3.5}$$

This follows from the definition of the Y_i and the fact that

$$(z-\chi_{\mu+\lambda_i}(z))^{m_i} \in I^{m_i}_{\mu+\lambda_i}.$$

Equivalently the submodules Y_i may be written

$$Y_i = \{y \in Y | (\tilde{z}-\alpha_i)^{m_i} y = 0\}, \tag{3.6}$$

(cf Kostant 1975). Thus Y_i is the unique maximal submodule of Y on which $(\tilde{z}-\alpha_i)$ is nilpotent.

It is our aim now to apply the polynomial identity (3.3) to construct projection operators which project the tensor product space $Y = V(\lambda) \otimes V$ onto the primary submodules Y_i. Such projectors are natural generalisations of the projection operators appearing in Bracken and Green (1971) (see also Gould 1980).

Following the classical theory of matrices (Gantmacher 1959) we begin by introducing the operators

$$Q_i = q_i(\tilde{z}) \tag{3.7}$$

where $q_i(x)$ denotes the polynomial

$$q_i(x) = \prod_{\substack{j=1 \\ \neq i}}^{n} \left(\frac{x-\alpha_j}{\alpha_i - \alpha_j}\right)^{m_j}. \tag{3.8}$$

Note, due to our assumptions about z, the numbers α_i $(i=1, \ldots, n)$ are all distinct so that Q_i is well defined. The operators (3.7), in the case where V is finite dimensional, would in themselves serve as projection operators. However, in infinite dimensions we need to consider the additional operators

$$H_i = h_i(\tilde{z})$$

where $h_i(x)$ denotes the polynomial

$$h_i(x) = \sum_{j=0}^{m_i-1} \xi_{i,j}(x-\alpha_i)^j \tag{3.9}$$

where the coefficients $\xi_{i,j}$ are given by

$$\xi_{i,j} = \frac{1}{j!}\left[\frac{1}{q_i(x)}\right]^{(j)}\Bigg|_{x=\alpha_i}, \qquad j \geq 0$$

and where $f^{(j)}(x)$ denotes the jth derivative $f(x)$ with $f^{(0)}(x) = f(x)$.

It is our aim to show that the required projection operators are given by the operators

$$P_i = H_i Q_i. \tag{3.10}$$

As a consequence of the polynomial identity (3.3) one may use a simple induction argument to establish the following result.

Lemma 1.

(a) Let $f(x)$ be an arbitrary polynomial. We then have

$$f(\tilde{z})(\tilde{z}-\alpha_i)^{m_i-l}Q_i = \sum_{j=0}^{l-1} \frac{f^{(j)}(\alpha_i)}{j!}(\tilde{z}-\alpha_i)^{m_i+j-l}Q_i \qquad \text{for } l = 1, \ldots, m_i$$

where

$$f^{(j)}(\alpha_i) = f^{(j)}(x)|_{x=\alpha_i}.$$

(b) Let $p_i(x) = h_i(x)q_i(x)$ where $q_i(x)$ and $h_i(x)$ denote the polynomials (3.8) and (3.9) respectively. Then

$$p_i^{(m)}(\alpha_i) = 0 \qquad \text{for } m_i > m \geq 1.$$

Our main result is

Theorem 1.

(a) The operators P_i, as defined by equations (3.7)–(3.10), satisfy the projection property

$$P_i P_j = \delta_{ij} P_j.$$

(b) On the space $Y = V(\lambda) \otimes V$ we have the following resolution of the identity

$$I = \sum_{i=1}^{n} P_i. \tag{3.11}$$

(c) The primary submodules Y_i occurring in the decomposition (3.4) are given by

$$Y_i = P_i Y.$$

Proof

(a) From Lemma 1(a) one has (putting $l = m_i$ and $f(x) = p_i(x)$)

$$P_i Q_i = \sum_{j=0}^{m_i-1} \frac{p_i^{(j)}(\alpha_i)}{j!}(z-\alpha_i)^j Q_i.$$

From Lemma 1(b) we have $p_i^{(j)}(\alpha_i) = 0$ for $j \geq 1$, whence

$$P_i Q_i = p_i(\alpha_i) Q_i = Q_i.$$

But $P_i = H_i Q_i$ whence

$$P_i^2 = P_i H_i Q_i = H_i P_i Q_i = H_i Q_i = P_i.$$

Using the polynomial identity (3.3) it is clear that $P_iP_j = 0$ for $i \neq j$. This completes the proof of part (a).

(b) Set $E = I - \Sigma_{i=1}^n P_i$. Using part (a) above one easily sees that E is a projection (i.e. $E^2 = E$) which satisfies $P_iE = 0$ $(i = 1, \ldots, n)$. Now put $W = EY$. Then W is a submodule of Y satisfying

$$P_iW = (0) \qquad i = 1, \ldots, n. \tag{3.12}$$

Now put $W_i = (\tilde{z} - \alpha_i)W \subseteq W$ and consider the factor module

$$M_i = W/W_i.$$

By definition $(\tilde{z} - \alpha_i)M_i = (0)$, which implies that $P_iM_i = M_i$. On the other hand, equation (3.12) implies that $P_iM_i = (0)$ which can only occur if $M_i = (0)$ whence we necessarily have

$$W = (\tilde{z} - \alpha_i)W \qquad i = 1, \ldots, n.$$

By repeated application of this result we must have

$$W = (\tilde{z} - \alpha_i)^{m_i}W \qquad i = 1, \ldots, n$$

whence

$$W = \prod_{i=1}^n (\tilde{z} - \alpha_i)^{m_i}W = (0),$$

where we have used the polynomial identity (3.3). Thus we have established that $EY = W = (0)$ which proves part (b).

(c) Part (c) is an easy consequence of the resolution (3.4) and the definition of the projection operators P_i.

Note that Theorem 1(a, b) was proved independently of the decomposition (3.4), so we may use this method of proof to give an independent derivation of (3.4).

In constructing the projection operators P_i we made use of the polynomial identity (3.3) which can be obtained solely from a knowledge of the infinitesimal character χ_μ of the characteristic module V. However, one may simplify the construction of the P_i if one knows the minimum polynomial identity

$$\prod_{i=1}^n (\tilde{z} - \alpha_i)^{n_i} = 0, \qquad n_i \leq m_i,$$

(see remarks following equation (3.4)) where n_i is the characteristic length of the primary submodule Y_i. In such a case the projectors P_i (acting on $Y = V(\lambda) \otimes V$) reduce to a simpler form given by the same construction as before but with m_i replaced by n_i. Clearly such a construction would involve more information than simply the infinitesimal character of the space V. In cases where such information is not available one may work with the projection operators (3.10).

For example, suppose $V = V(\mu)$ were a finite dimensional irreducible representation with highest weight $\mu \in \Lambda^+$. Then in this case (using the same notation as before) $\tilde{z}$ satisfies the identity

$$\prod_{i=1}^n (\tilde{z} - \alpha_i) = 0. \tag{3.13}$$

(In fact only factors corresponding to $\mu+\lambda_i \in \Lambda^+$ need to be retained in this identity.) Our required projection operators in this case are

$$P_i' = \prod_{j\neq i}\left(\frac{z-\alpha_j}{\alpha_i-\alpha_j}\right), \qquad P_i'P_j' = \delta_{ij}P_j'. \tag{3.14a, b}$$

Substituting equations (3.13) and (3.14) into (3.10) one sees that the projection operators defined by (3.10) reduce to the form (3.14*a*).

We conclude this section by obtaining a generalisation of the identity resolution (3.11). Suppose $f(x)$ is an arbitrary function (expandable in a power series). Then Lemma 1(a) implies (putting $l=m_i$)

$$f(\tilde{z})Q_i = \sum_{j=0}^{m_i-1}\frac{f^{(j)}(\alpha_i)}{j!}(\tilde{z}-\alpha_i)^jQ_i,$$

whence, using $P_i = H_iQ_i = Q_iH_i$,

$$f(\tilde{z})P_i = \sum_{j=0}^{m_i-1}\frac{f^{(j)}(\alpha_i)}{j!}(\tilde{z}-\alpha_i)^jP_i.$$

Hence as a generalisation of the spectral resolution (3.11) we obtain

$$f(\tilde{z}) = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\frac{f^{(j)}(\alpha_i)}{j!}(\tilde{z}-\alpha_i)^jP_i, \tag{3.15}$$

where $f(x)$ is an arbitrary function. In particular we may define an inverse of the operator $\tilde{z}$ by setting

$$\tilde{z}^{-1} = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1}(-1)^j\frac{(\tilde{z}-\alpha_i)^j}{\alpha_i^{j+1}}P_i,$$

which is well defined provided $\alpha_i \neq 0$ $(i=1,\ldots,n)$.

The nicest case occurs when the infinitesimal characters $\chi_{\mu+\lambda_i}$ $(i=1,\ldots,k)$ are all distinct (i.e. $n=k$). In such a case the projection operators P_i reduce to the simpler form

$$P_i = \prod_{\substack{j=1\\ \neq i}}^{k}\left(\frac{\tilde{z}-\alpha_j}{\alpha_i-\alpha_j}\right)$$

and (3.15) reduces to

$$f(\tilde{z}) = \sum_{i=1}^{k} f(\alpha_i)P_i.$$

In this case the tensor product representation $Y = V(\lambda)\otimes V$ decomposes into a direct sum of characteristic submodules

$$Y = \bigoplus_{i=1}^{k} Y_i \tag{3.16}$$

where

$$\begin{aligned} Y_i &= \{y\in Y \mid uy = \chi_{\mu+\lambda_i}(u)y, \qquad \text{for all } u\in Z\}\\ &= \{y \in Y \mid (\tilde{z}-\alpha_i)y = 0\}\end{aligned}$$

is the unique maximal submodule of y admitting the infinitesimal character $\chi_{\mu+\lambda_i}$.

4. Shift tensors

Following our previous notation let $V(\lambda)$ be a finite dimensional irreducible representation of L and let $\lambda_1, \ldots, \lambda_k$ be the distinct weights occurring in $V(\lambda)$. Choose an orthonormal basis $e_1, \ldots, e_d$ ($d = \dim V(\lambda)$) for $V(\lambda)$. With respect to this basis we define an irreducible tensor operator of rank λ as a collection of operators $\{T_\alpha\}_{\alpha=1}^d$ which transform according to

$$[x, T_\alpha] = \sum_{\beta=1}^{d} T_\beta \pi_\lambda(x)_{\beta\alpha}, \qquad x \in L.$$

Associated with the tensor operator T_α is its domain V and range W i.e.

$$T_\alpha v \in W \qquad \text{for all } v \in V; \qquad \alpha = 1, \ldots, d,$$

where V and W are (possibly infinite dimensional) U-modules. This is equivalent to investigating the action of an intertwining operator

$$T: V(\lambda) \otimes V \to W \tag{4.1}$$

$$\pi_W(x) T = T(\pi_\lambda(x) \otimes 1 + \pi_V(x)), \qquad x \in L$$

where π_V (resp. π_W) is the representation afforded by V (resp. W). In other words

$$T \in \mathrm{Hom}_L(V(\lambda) \otimes V, W)$$

is an element of the set of all operators from $V(\lambda) \otimes V$ to W commuting with the action of L (and hence U). We recover our usual component form definition of a tensor operator by setting

$$T_\alpha v = T(e_\alpha \otimes v), \qquad \alpha = 1, \ldots, d, \qquad v \in V.$$

For notational convenience we follow the notation of § 3 and denote the tensor product space $V(\lambda) \otimes V$ simply by Y. Without loss of generality we assume that $W = TY$ i.e. that the mapping (4.1) is onto. In such a case the isomorphism theorem guarantees that W is isomorphic to the factor module $Y/\ker T$, where $\ker T$ is the submodule of Y given by

$$\ker T = \{y \in Y \mid Ty = 0\}.$$

Note that because T commutes with the L action one sees that $\ker T$ is a U-submodule of Y. Thus we have reduced our problem to investigating the structure of the spaces $\mathrm{Hom}_L(Y, Y/Y_1)$, where Y_1 is an arbitrary submodule of Y.

We wish now to apply our previous results to investigate the action of a tensor operator T_α on an arbitrary characteristic representation V with characteristic χ_μ, $\mu \in H^*$. Following the notation of § 3 let $n(\leq k)$ be the number of distinct infinitesimal characters occurring in the set $\{\chi_{\mu+\lambda_i}\}_{i=1}^k$ and suppose the λ_i are ordered so that the infinitesimal characters $\chi_{\mu+\lambda_1}, \ldots, \chi_{\mu+\lambda_n}$ are distinct. Finally choose $z \in Z$ such that the numbers $\chi_{\mu+\lambda_1}(z), \ldots, \chi_{\mu+\lambda_n}(z)$ are all distinct.

Our previous analysis shows that on the space $Y = V(\lambda) \otimes V$ we have the following resolution of the identity

$$I = \sum_{i=1}^{n} P_i, \qquad P_i P_j = \delta_{ij} P_j$$

where the projectors P_i are given by equation (3.10). This implies a decomposition of the space Y into a direct sum of primary submodules (cf equation (3.4))

$$Y = \bigoplus_{i=1}^{n} Y_i.$$

Hence we may resolve the intertwining operator T into components

$$T = \sum_{i=1}^{n} T[i] \tag{4.2}$$

where $T[i] = TP_i$. Then, by definition $T[i]Y = TP_iY = TY_i$. Thus since T intertwines the action of U, this implies that $W = TY$ decomposes into a direct sum of submodules

$$W = \bigoplus_{i=1}^{n} W_i$$

where $W_i = TY_i$ is a primary submodule of W with characteristic $\chi_{\mu+\lambda_i}$. Thus $T[i]$ is an intertwining operator from Y to the primary module W_i.

In component form this implies that the tensor operator T_α may be resolved into generalised shift components

$$T_\alpha = \sum_{i=1}^{n} T[i]_\alpha \tag{4.3}$$

where $T[i]_\alpha$ is given by

$$T[i]_\alpha v = TP_i(e_\alpha \otimes v), \qquad \alpha = 1, \ldots, d, \qquad v \in V.$$

Since $T[i] = TP_i$ is an intertwining operator we note that $\{T[i]_\alpha\}_{\alpha=1}^{d}$ also constitutes an irreducible tensor operator of rank λ (with domain V and range W_i).

Thus we have shown that a tensor operator T_α when acting on a characteristic module V of characteristic χ_μ may be resolved into generalised shift components $T[i]_\alpha$ where $T[i]_\alpha$ takes V to a primary representation of characteristic $\chi_{\mu+\lambda_i}$. Thus $T[i]_\alpha$ shifts characteristics of representations from χ_μ to $\chi_{\mu+\lambda_i}$. In finite dimensions this is equivalent to shifting the highest weight.

Note that the projection operator P_i itself is an intertwining operator $P_i: Y \to Y_i$ and hence determines a tensor operator $\{W[i]_\alpha\}$ defined by

$$W[i]_\alpha v = P_i(e_\alpha \otimes v), \qquad \alpha = 1, \ldots, d, \qquad v \in V.$$

The tensor operators $\{W[i]_\alpha\}_{\alpha=1}^{d}$ are generalisations of the unit Wigner operators considered by Biedenharn and Louck (1972) and Louck and Biederharn (1970).

Following the infinitesimal techniques introduced in Gould (1980, 1981) (see also Bracken and Green 1971, Green 1971) we note that given an irreducible tensor operator T_α acting on the characteristic representation V one may construct the generalised shift components $T[i]_\alpha$ using operators in the universal enveloping algebra U. This follows from the observation (see Gould 1982) that $\tilde{z}$ (notation as in § 3) may be viewed as a $d \times d$ matrix with entries $\tilde{z}_{\alpha\beta} \in U$ ($\alpha, \beta = 1, \ldots, d = \dim V(\lambda)$). As an illustration consider the case where $z = C_L$ is the universal Casimir element. In this case we may write

$$\begin{aligned}\tilde{C}_L &= -\tfrac{1}{2}[\partial(C_L) - \pi_\lambda(C_L) \otimes 1 - 1 \otimes C_L] \\ &= -\sum_r \pi_\lambda(x^r)x_r \end{aligned}\tag{4.4}$$

where $\{x_r\}$ is a basis of L and $\{x^r\}$ is the corresponding dual basis with respect to the Killing form. In this case we have

$$[\tilde{C}_L]_{\alpha\beta} = -\sum_r \pi_\lambda(x^r)_{\alpha\beta} x_r \qquad \alpha, \beta = 1, \ldots, d$$

where $\pi_\lambda(x)(x \in L)$ is the matrix representing x in the basis $\{e_\alpha\}$ of $V(\lambda)$. This is the matrix considered by Bracken and Green (1971) and Green (1971) for the classical Lie groups.

The advantage of this procedure is that the $d \times d$ matrix $\tilde{z}$ is defined in a representation-independent way as a matrix over the universal enveloping algebra U. Polynomials in the matrix $\tilde{z}$ may then be defined recursively according to

$$[\tilde{z}^m]_{\alpha\beta} = \sum_{\gamma=1}^{d} \tilde{z}_{\alpha\gamma}[\tilde{z}^{m-1}]_{\gamma\beta} = \sum_{\gamma=1}^{d} [\tilde{z}^{m-1}]_{\alpha\gamma}\tilde{z}_{\gamma\beta} \in U.$$

Acting on the representation V we may thus regard the operator $\tilde{z}$ as a $d \times d$ matrix with entries $\tilde{z}_{\alpha\beta} \in \pi_V(U)$. Then, since the projector P_i (see equations (3.8)–(3.10)) is a polynomial in the matrix $\tilde{z}$ we may also regard P_i as a $d \times d$ matrix with entries $[P_i]_{\alpha\beta} \in \pi_V(U)$. With this convention the generalised shift components $T[i]_\alpha$ of the tensor operator T_α are given in component form explicitly by

$$T[i]_\alpha = \sum_{\beta=1}^{d} T_\beta [P_i]_{\beta\alpha}.$$

The nicest case occurs when the infinitesimal characters $\chi_{\mu+\lambda_i}$ $(i = 1, \ldots, k)$ are all distinct (i.e. $n = k$). In this case the projection operators are given simply by (notation as in § 3)

$$P_i = \prod_{j \neq i} \left(\frac{\tilde{z} - \alpha_j}{\alpha_i - \alpha_j} \right)$$

and the space $Y = V(\lambda) \otimes V$ decomposes into a direct sum of characteristic submodules (see equation (3.16)). The shift component $T[i]_\alpha$ of the tensor operator T_α in this case takes the characteristic module V to the characteristic module $W_i = TY_i = TP_iY$ (notation as before) with characteristic $\chi_{\mu+\lambda_i}$ i.e. $T[i]_\alpha$ shifts to the infinitesimal character from χ_μ to $\chi_{\mu+\lambda_i}$.

In applications one frequently has to deal with contragredient tensor operators $\tilde{T}_\alpha$ (with domain V and range $\tilde{W}$) which transform according to

$$[x, \tilde{T}_\alpha] = -\sum_{\beta=1}^{d} \tilde{T}_\beta \pi_\lambda^T(x)_{\beta\alpha}, \qquad x \in L.$$

In this case we follow the same procedure as before except that we consider the tensor product space $V^*(\lambda) \otimes V$ where $V^*(\lambda)$ is contragredient to $V(\lambda)$. (Note that the weights occurring in $V^*(\lambda)$ are the negative of those occurring in $V(\lambda)$ from which it follows that the infinitesimal characters occurring in the space $V^*(\lambda) \otimes V$ are of the form $\chi_{\mu-\lambda_i}(i = 1, \ldots, k)$). An important example where a tensor operator and its contragredient appear will be considered in the following section.

5. Example: vector operators for gl(n, $\mathbb{C}$)

We consider the case where L is the Lie algebra of gl(n, $\mathbb{C}$) which is n^2 dimensional with basis $a_{ij}(i, j = 1, \ldots, n)$ satisfying the commutation relations

$$[a_{ij}, a_{kl}] = \delta_{kj} a_{il} - \delta_{il} a_{kj}.$$

As a particular case of our previous results we aim to show that the work of Green (1971) extends also to infinite dimensional representations. We shall also briefly discuss the application of these methods to determining the matrix elements of the U(p, q) generators in the discrete series of unitary representations. Further work along these lines is now in progress.

Note that gl(n, $\mathbb{C}$) is not semi-simple but is reductive and all of our previous results apply. We have in fact the following decomposition of ideals

$$\mathrm{gl}(n, \mathbb{C}) = \mathrm{sl}(n, \mathbb{C}) \oplus \mathbb{C} I_1,$$

where sl(n, $\mathbb{C}$) is a semi-simple Lie algebra and

$$I_1 = \sum_{i=1}^{n} a_{ii}$$

is an invariant (i.e. commutes with all the gl(n, $\mathbb{C}$) generators). Since we only consider representations of gl(n, $\mathbb{C}$) on which I_1 acts as a scalar, one sees that the representation theory of gl(n, $\mathbb{C}$) and sl(n, $\mathbb{C}$) are equivalent.

Throughout we denote the Lie algebra gl(n, $\mathbb{C}$) simply by L. Now we take our Cartan subalgebra $H \subseteq L$ to be the n-dimensional sub-space of L spanned by the diagonal generators a_{ii}. If $\lambda \in H^*$ is a weight of L we identify λ with the n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_i = \lambda(a_{ii})$.

We take as our set of positive roots Φ^+ of L, the weights $\Delta_i - \Delta_j$ $(i < j)$ where Δ_i is the weight with 1 in the ith position and zeros elsewhere. In this case δ (the half-sum of the positive roots) is given by

$$\delta = \tfrac{1}{2} \sum_{i<j} (\Delta_i - \Delta_j) = \tfrac{1}{2} \sum_{r=1}^{n} (n + 1 - 2r) \Delta_r.$$

The Weyl group of course is the symmetric group S_n (i.e. the group of permutations on n objects).

The centre Z of the universal enveloping algebra U of L is generated by the algebraically independent Gel'fand invariants

$$I_r = \mathrm{Tr}(a^r) = \sum_{i=1}^{n} a_{ii}^r \qquad r = 1, \ldots, n$$

where we define

$$a_{ij}^m = \sum_{k=1}^{n} a_{ik} a_{kj}^{m-1} = \sum_{k=1}^{n} a_{ik}^{m-1} a_{kj}.$$

Note that the second-order invariant I_2 determines the $\tilde{W}$-invariant polynomial function defined by

$$\chi_\lambda(I_2) = (\lambda, \lambda + 2\delta) = \sum_{r=1}^{n} \lambda_r(\lambda_r + n + 1 - 2r)$$

where we define

$$(\lambda, \mu) = \sum_{r=1}^{n} \lambda_r \mu_r, \qquad \lambda, \mu \in H^*.$$

In the case where $\pi_\lambda = \pi^*$ is the fundamental contragredient vector representation of L the matrix $\tilde{C}_L$ in equation (4.4) reduces to

$$\tilde{C}_L = -\sum_{i,j=1}^{n} \pi^*(a_{ji}) a_{ij} = \sum_{i,j=1}^{n} E_{ij} a_{ij}$$

where E_{ij} is a typical elementary matrix with 1 in the (i, j) position and zeros elsewhere. This is the matrix considered by Green (1971) for $\mathrm{GL}(n)$. Following the notation of Green (1971) we denote the matrix $\tilde{C}_L$ by a.

Note that the weights occurring in the U-module V^* (where V^* is the representation space of π^*) are of the form $-\Delta_i (i = 1, \ldots, n)$ each occurring with multiplicity 1.

Now let W be any characteristic U-module admitting an infinitesimal character χ_μ, $\mu \in H^*$. We further assume that

$$(\mu + \delta, \alpha) \neq 0 \qquad \text{for all } \alpha \in \Phi^+, \tag{5.1}$$

(this assumption is always satisfied when μ is dominant integral). It is customary to call weights μ satisfying equation (5.1) regular. With this condition it is clear that the infinitesimal characters $\chi_{\mu-\Delta_i} (i = 1, \ldots, n)$ (and $\chi_{\mu+\Delta_i}$, $i = 1, \ldots, n$) are all distinct. In fact we have

$$\begin{aligned}
\chi_{\mu-\Delta_j}(I_2) &- \chi_{\mu-\Delta_i}(I_2) \\
&= (\mu - \Delta_j, \mu - \Delta_j + 2\delta) - (\mu - \Delta_i, \mu - \Delta_i + 2\delta) \\
&= 2(\Delta_i - \Delta_j, \mu + \delta) = \chi_{\mu+\Delta_i}(I_2) - \chi_{\mu+\Delta_j}(I_2).
\end{aligned}$$

Assuming (5.1) is satisfied this shows that

$$\chi_{\mu-\Delta_j}(I_2) \neq \chi_{\mu-\Delta_i}(I_2), \qquad \chi_{\mu+\Delta_j}(I_2) \neq \chi_{\mu+\Delta_i}(I_2) \qquad \text{for } i \neq j.$$

From the remarks of § 3 (see also Gould 1982) it follows that on the space W, a satisfies the polynomial identity

$$\prod_{r=1}^{n} (a - \alpha_r) = 0,$$

where $\alpha_r = \mu_r + n - r$. It is useful in this case to regard the roots α_r as operators lying in an algebraic extension of the centre Z (Gould 1982). The roots α_r take constant values $\chi_\mu(\alpha_r) = \mu_r + n - r$ on a representation admitting χ_μ as an infinitesimal character.

The projection operators

$$P_r = \prod_{l \neq r} \left(\frac{a - \alpha_l}{\alpha_r - \alpha_l} \right)$$

project the tensor product space $V^* \otimes W$ onto a characteristic module $W_r = P_r(V^* \otimes W)$ with infinitesimal character $\chi_{\mu-\Delta_r}$. Note that the projectors P_r are well defined, since the α_r are all distinct (in view of equation (5.1)), and satisfy

$$I = \sum_{r=1}^{n} P_r, \qquad P_i P_j = \delta_{ij} P_j.$$

If $\{\tilde{\psi}_i\}_{i=1}^n$ is a contragradient vector operator of L, i.e. $\tilde{\psi}$ satisfies $[a_{ij}, \tilde{\psi}_k] = -\delta_{ik}\tilde{\psi}_j$, then $\tilde{\psi}_i$ may be resolved into (shift) components $\tilde{\psi}[r]_i = (\tilde{\psi}P_r)_i$ where each $\tilde{\psi}[r]$ is an intertwining operator from $V^* \otimes W$ onto W_r (or at least a quotient of W_r). Thus $\tilde{\psi}[r]_i$ shifts the infinitesimal character of W from χ_μ to $\chi_{\mu-\Delta_r}$.

Similarly we may consider the adjoint matrix $\tilde{a}$ of a defined by $\tilde{a}_{ij} = -a_{ji}$. In this case $\tilde{a}$ satisfies the polynomial identity

$$\prod_{r=1}^{n} (\tilde{a} - \tilde{\alpha}_r) = 0,$$

where $\tilde{\alpha}_r = n - 1 - \alpha_r$. Our required projection operators in this case are

$$\tilde{P}_r = \prod_{l \neq r} \left(\frac{\tilde{a} - \tilde{\alpha}_l}{\tilde{\alpha}_r - \tilde{\alpha}_l} \right)$$

which project the tensor product space $V \otimes W$ (where V is the representation space for the fundamental vector representation) onto the characteristic submodule $\tilde{W}_r = \tilde{P}_r(V \otimes W)$ with infinitesimal character $\chi_{\mu+\Delta_r}$. As for the P_r one sees that the projection operators $\tilde{P}_r$ are well defined and satisfy the relations

$$I = \sum_{r=1}^{n} \tilde{P}_r, \qquad \tilde{P}_i \tilde{P}_j = \delta_{ij} \tilde{P}_j.$$

If $[\psi_i]_{i=1}^n$ is a vector operator of L, i.e. ψ satisfies $[a_{ij}, \psi_k] = \delta_{jk}\psi_i$, then ψ_i may be resolved into shift components $\psi[r]_i = (\psi\tilde{P}_r)_i$ where each $\psi[r]_i$ shifts the infinitesimal character of W from χ_μ to $\chi_{\mu+\Delta_r}$.

Consider for example the subgroup imbedding $\mathrm{GL}(n+1) \supset \mathrm{GL}(n)$. Let $\tilde{V}$ be an (irreducible) characteristic representation of $\mathrm{GL}(n+1)$ which admits an infinitesimal character $\tilde{\chi}_\mu$ (where μ is a weight of $\mathrm{GL}(n+1)$). Suppose $\tilde{V}$ decomposes into a direct sum of irreducible characteristic $\mathrm{GL}(n)$-submodules;

$$\tilde{V} = \bigoplus_{\nu} V_\nu, \tag{5.2}$$

where V_ν is irreducible and admits the infinitesimal character χ_ν, $\nu \in H^*$. Suppose further that the infinitesimal characters χ_ν occurring in (5.2) all occur with unit multiplicity.

Now the $\mathrm{GL}(n+1)$ generators $\psi_i = a_{i,n+1}$ $(i = 1, \ldots, n)$ constitute a vector operator of $\mathrm{GL}(n)$. If $\psi[r]_i$ denote the shift components of ψ_i our analysis shows that

$$\psi[r] V \otimes V_\nu = V_{\nu+\Delta_r}, \tag{5.3}$$

where it is understood that the RHS of (5.3) is zero if $V_{\nu+\Delta_r}$ does not occur in the decomposition (5.2). Note also that if we relax the assumption that V_ν is irreducible, to V_ν is completely reducible, then (5.3) relaxes to $\psi[r] V \otimes V_\nu \subseteq V_{\nu+\Delta_r}$.

Equation (5.3) is to be understood in the sense that if $v \in V_\nu$ then

$$\psi[r]_i v = \sum_{j=1}^{n} a_{j,n+1} \left[\prod_{l \neq r} \left(\frac{\tilde{a} + \nu_l - l + 1}{\nu_l - \nu_r + r - l} \right) \right]_{ji} v \in V_{\nu+\Delta_r}.$$

Conversely, given $v \in V_{\nu+\Delta_r}$ there exist vectors $v_i \in V_\nu (i = 1, \ldots, n)$ such that

$$v = \sum_{i=1}^{n} \psi[r]_i v_i.$$

Similarly the GL$(n+1)$ generators $\tilde{\psi}_i = a_{n+1,i}$ $(i=1,\ldots,n)$ constitute a contragradient vector operator. If $\tilde{\psi}[r] = \tilde{\psi}P_r$ denote the shift components of $\tilde{\psi}$ we have

$$\tilde{\psi}[r]V^* \otimes V_\nu = V_{\nu - \Delta_r}.$$

Since the infinitesimal character of a representation is uniquely determined by the eigenvalues of the Gel'fand invariants (and vice versa) our work shows that, in the case considered above, the generators $a_{i,n+1}$ (and $a_{n+1,i}$) shift the eigenvalues of the Gel'fand invariants in exact analogy with the finite dimensional case. This is precisely the situation which occurs in the discrete series of unitary representations of U$(p, q+1)$ $(n = p+q)$ which decompose into unitary characteristic irreducible representations of U(p, q). In fact a Gel'fand-Zetlin type basis exists for these representations (see Chakrabarti (1968) which are labelled by the eigenvalues of the Gel'fand invariants for each of the subgroups occurring in the chain

$$\mathrm{U}(p, q+1) \supset \mathrm{U}(p, q) \supset \mathrm{U}(p, q-1) \supset \ldots \supset \mathrm{U}(p) \supset \ldots \supset \mathrm{U}(1). \tag{5.4}$$

Our analysis shows that the generators of U$(p, q+1)$ shift the labels of the Gel'fand states (or equivalently the eigenvalues of the Gel'fand invariants for each subgroup occurring in the chain (5.4)) in exact analogy with the finite dimensional case. We may then proceed to evaluate the matrix elements of the U$(p, q+1)$ generators directly in this basis, using the methods of Gould (1980, 1981). This approach offers an alternative to the usual method whereby the matrix elements for the discrete series of unitary representations of U(p, q) are obtained from the compact case U$(n = p+q)$ using the methods of analytic continuation. A similar analysis may be applied to the pseudo-orthogonal groups O(p, q).

It would be of interest to extend these results to the action of irreducible tensor operators on an irreducible Harish-Chandra module V (see Dixmier 1977). In this case Kostant (1978) shows that the space $V(\lambda) \otimes V$ admits a finite composition series (see Curtis and Reiner 1962) and as such decomposes uniquely into a finite direct sum of indecomposable submodules (Krull–Schmidt Theorem—see Curtis and Reiner (1962)). In the case where V is a submodule of a representation $\hat{V}$ realised on a Hilbert space and T_α is a tensor operator with domain $\hat{V}$ and range $W \subseteq \hat{V}$, then a suitable generalisation of the Wigner–Eckart theorem may be obtained.

We hope to look at the above mentioned problems in a future publication.

References

Baird G E and Biedenharn L C 1964 *J. Math. Phys.* **5** 1723

Biedenharn L C and Louck J D 1972 *J. Math. Phys.* **13** 1985

Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *J. Funct. Anal.* **5** 1

Bracken A J and Green H S 1971 *J. Math. Phys.* **12** 2099

Curtis C W and Reiner I 1962 *Representation Theory of Finite Groups and Associative Algebras* (New York: Wiley)

Chakrabarti A 1968 *J. Math. Phys.* **9** 2087

Dixmier J 1977 *Enveloping Algebras* (Amsterdam: North-Holland)

Edwards S A and Gould M D 1982 *Cyclic Vectors, Projection Methods and the Multiplicity Problem* University of Adelaide preprint (to appear in *J. Math. Phys.*)

Fano U 1962 *Eastern Theoretical Physics Conference* (unpublished)

Gantmacher F R 1959 *The Theory of Matrices* vol 1, 2 (New York: Chelsea)

Gould M D 1980 *J. Math. Phys.* **21** 444
——1981 *J. Math. Phys.* **22** 2376
——1982 *Characteristic Identities for Semi-simple Lie Algebras*, University of Adelaide preprint (to appear in *J. Aust. Math. Soc.* B)
Green H S 1971 *J. Math. Phys.* **12** 2106
Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (Berlin: Springer-Verlag)
Jarvis P D and Green H S 1979 *J. Math. Phys.* **20** 2115
Klimyk A U 1977 *Math. Phys Lett.* **1** 375
Kostant B 1975 *J. Funct. Anal.* **20** 257
Louck J D and Biedenharn L C 1970 *J. Math. Phys.* **11** 2368
Zuckerman G 1977 *Ann. Math.* **106** 295